Int. J. Heat Mass Transfer. Vol. 15, pp. 733–740. Pergamon Press 1972. Printed in Great Britain

APPLICATION OF THE INTEGRAL METHOD TO FLOWS WITH AXIAL DIFFUSION

YEHUDA TAITEL
School of Engineering, Tel-Aviv University, Ramat Aviv, Israel

and

ABRAHAM TAMIR
School of Engineering, University of the Negev, Beer-Sheva, Israel

(*Received* 18 *March* 1971 *and in revised form* 22 *July* 1971)

Abstract—This work extends the application of the integral method to problems with axial diffusion usually being ignored in convective flow problems. It is demonstrated that heat (or mass)-transfer characteristics can be obtained in a closed form fashion and with satisfactory accuracy. Results for the Graetz problem and other problems with axial diffusion are reported.

NOMENCLATURE

A, cross sectional flow area;
c_1, constant, Table 1;
c_2, constant, Table 1;
c_3, constant, Table 1;
c_p, heat capacity at constant pressure;
d, pipe diameter;
h, local heat-transfer coefficient;
$\bar{h}$, average heat-transfer coefficient;
k, thermal conductivity;
l, channel depth or pipe radius;
$\overline{Nu}_l$, average Nusselt number, $\bar{h}l/k$;
$\overline{Nu}_d$, average Nusselt number, $\bar{h}d/k$;
Pe, Péclét number, $\rho c_p \bar{u} 2l/k$;
q, heat flux;
Q, net rate of heat flow;
S, heat-transfer area;
Sh, Sherwood number;
T, temperature;
$\bar{T}$, mean temperature, $\int_0^l Tu\,\mathrm{d}y/l\bar{u}$;
u, velocity;
$\bar{u}$, mean velocity, $\int_0^l u\,\mathrm{d}y/l$;
x, axis along streamlines (Fig. 1);
y, axis perpendicular to streamlines (Fig. 1);
α, eigenvalue;
δ, thermal boundary-layer thickness;
η, dimensionless coordinate, y/l;
θ, dimensionless temperature;
$\bar{\theta}$, average dimensionless temperature;
λ, eigen value;
ξ, dimensionless coordinate, x/l;
ρ, density.

Subscripts and superscripts
$'$, primed coordinate system, Appendix A;
δ, edge of boundary layer;
l, at insulated surface or centre line;
n, running index;
0, at inlet;
w, at thermal disturbing wall.

INTRODUCTION

THIS work is concerned with the application of of the "integral method" to problems of convection with axial diffusion.

The integral method is a well-known approximate technique for obtaining solutions to problems otherwise difficult to obtain. Its advantage lies in its simplicity in converting complex and non-linear problems into tractable form often obtainable in a closed form fashion.

The integral method has been traditionally used for flow, mass- and heat-transfer problems to the extent that almost any standard book in the field of heat and mass transfer of fluid mechanics details this method. It seems, however, that the use of this method was conclusively limited to problems of convection with negligible axial diffusion. The purpose of this work is to extend the use of the integral method to problems in which axial diffusion is not negligible.

The question whether axial diffusion (i.e. diffusion in the main flow direction) can be neglected, is usually characterized by the Péclét number. In most conventional engineering problems the Péclét number is high and the neglect of axial diffusion is considered safe. Recently, however, there has been growing interest in flows of small Péclét numbers to the extent that axial diffusion may be appreciable. This include cases of diffusion of oxygen in blood flow [1], dispersion of solute in Poiseulle pipe flow [2] and other cases [3, 4].

Except for the Graetz problem all other problems considered here have an exact solution. Indeed, they serve as a guide for reflecting upon the accuracy of the integral technique in this case.

ANALYSIS

We consider the case of flow in a channel or in a pipe. The flow is in the x direction and the velocity profile depends on y alone (see Fig. 1). The thermal disturbance is applied at $y = 0$ and $y = l$ is either a symmetry line or ideal insulator. In the case of pipe flow, l is the radius of the pipe and $y = l$ is the pipe axis.

The energy differential equation in this case, takes the form

$$\rho c_p u(y)\frac{\partial T}{\partial x} = k\left[\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}\right] \tag{1}$$

for the channel flow, and

$$\rho c_p u\frac{\partial T}{\partial x} = k\left[\frac{1}{l-y}\frac{\partial}{\partial y}(l-y)\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial x^2}\right] \tag{2}$$

for the pipe flow.

The thermal disturbance is applied on the plane $y = 0$ where $x > 0$. $\delta(x)$ represents the penetration of this thermal disturbance into the flowing stream. The integral equivalent energy equation is obtained by integrating equation (1) from $y = 0$ to $y = \delta$. For equation (2) this integration is preceded by multiplying equation (2) by $2\pi(l - y)$.

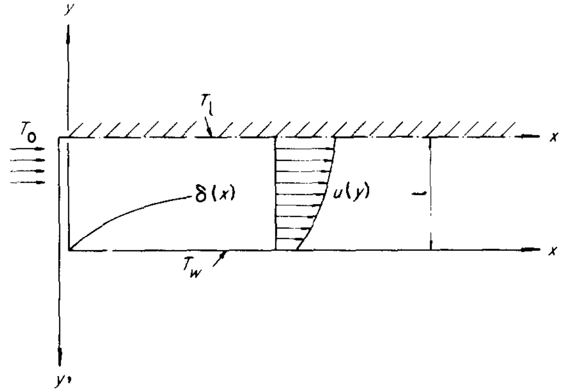

FIG. 1. Physical model and coordinate system.

The first two terms of equation (1) are handled in the usual manner, namely

$$\int_0^\delta u(y)\frac{\partial T}{\partial x}\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta uT\mathrm{d}y - \frac{\mathrm{d}\delta}{\mathrm{d}x}(uT)_\delta \tag{3}$$

$$\int_0^\delta \frac{\partial^2 T}{\partial y^2}\mathrm{d}y = \left.\frac{\partial T}{\partial y}\right|_\delta - \left.\frac{\partial T}{\partial y}\right|_w. \tag{4}$$

The third term, which contains the axial flow effect yields

$$\int_0^\delta \frac{\partial^2 T}{\partial x^2}\mathrm{d}y = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_0^\delta T\,\mathrm{d}y - \frac{\mathrm{d}}{\mathrm{d}x}\left(T_\delta\frac{\mathrm{d}\delta}{\mathrm{d}x}\right) - \frac{\mathrm{d}\delta}{\mathrm{d}x}\left.\frac{\partial T}{\partial x}\right|_\delta. \tag{5}$$

On δ, $T_\delta = T[x, \delta(x)]$ therefore

$$\frac{\mathrm{d}T_\delta}{\mathrm{d}x} = \left.\frac{\partial T}{\partial x}\right|_\delta + \left.\frac{\partial T}{\partial y}\right|_\delta\frac{\mathrm{d}\delta}{\mathrm{d}x} \tag{6}$$

Substituting equations (3–6) in (1) we obtain the integral energy equation in the form:

$$\rho c_p\left[\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u\,T\mathrm{d}y - \frac{\mathrm{d}\delta}{\mathrm{d}x}(uT)_\delta\right] = k\left\{\left.\frac{\partial T}{\partial y}\right|_\delta\right.$$

$$-\left.\frac{\partial T}{\partial y}\right|_w + \frac{d^2}{dx^2}\int_0^\delta T\,dy - \frac{d}{dx}\left(T_\delta \frac{d\delta}{dx}\right) - \frac{d\delta}{dx}\frac{dT_\delta}{dx} + \left(\frac{d\delta}{dx}\right)^2 \left.\frac{\partial T}{\partial y}\right|_\delta \Bigg\}. \quad (7)$$

Similarly equation (2) is transformed into:

$$\rho c_p\left[\frac{d}{dx}\int_0^\delta (l-y)uT\,dy - \frac{d\delta}{dx}\ (l-\delta)\,u_\delta T_\delta\right]$$

$$= k\Bigg\{(l-\delta)\left.\frac{\partial T}{\partial y}\right|_\delta - l\left.\frac{\partial T}{\partial y}\right|_w + \frac{d^2}{dx^2}\int_0^\delta (l-y)T\,dy - \frac{d}{dx}\left[(l-\delta)T_\delta \frac{d\delta}{dx}\right] - \frac{d\delta}{dx}(l-\delta)\frac{dT_\delta}{dx} + \left(\frac{d\delta}{dx}\right)^2 (l-\delta)\left.\frac{\partial T}{\partial y}\right|_\delta\Bigg\} \quad (8)$$

Taking note of the usual procedures for solving heat- or mass-transfer problems where axial diffusion is neglected it is revealed that:

(a) For a semi infinite flow ($l \to \infty$) equation (7) (without the axial diffusion terms) is usually solved for $\delta = \delta(x)$.
(b) For a channel flow, the above-mentioned treatment is applied up to the point where $\delta(x) = l$. Thereafter, δ remains equal to l and one looks for a new variable to be solved by equation (7) such as $q_w(x)$ or $T_l(x)$. $q_w(x)$ or $T_l(x)$ is matched to the previously obtained result in the region $\delta(x) < l$. The same applies to the case of axial symmetry (pipe flow) although treatment "a" for $\delta(x) < l$ is not synonymous to the "semi infinite" case as in the two dimensional one.

If we try to apply this procedure to the case with axial conduction we obtain a second order non-linear differential equation for δ which does not have (to the best of our knowledge) a closed form solution. Keeping in mind that the integral method is attractive only if it is amenable to an easy preferably closed form answer, we need further simplifications to achieve this aim.

The procedure adopted here is to ignore the conventional afore-mentioned two-step treatment and resort to step "b" alone. This may lead to error in the solution for very small x near the "entrance region" but we expect those of large x to be of reasonable accuracy. This approach yields quite simple closed form solutions to problems for which an exact analysis is either difficult or impossible.

In the present work we consider four cases of axial diffusion:

(1) Diffusion of gases to flow over an inclined plane assuming the ideal slug flow profile. This problem covers also heat transfer by direct contact condensation of vapour on a liquid film.
(2) The same as above assuming fully developed velocity profile.
(3) Heat transfer to pipe flow assuming slug flow.
(4) Heat transfer to pipe flow assuming fully developed velocity profile.

For all these problems, we have the following boundary conditions:

$$\begin{aligned} x &= 0 & T &= T_0 \\ x &\to \infty & T &= T_w\,(\text{or } \partial T/\partial x = 0) \\ y &= 0 & T &= T_w \\ y &= l & \partial T/\partial y &= 0. \end{aligned} \quad (9)$$

The exact solution to problem (2) has been recently presented by Rotem and Neilson [1]. Problems (1) and (3) can easily be derived exactly by the standard method of separation of variables as briefly summarized in Appendix A. Comparison of those available solutions with our integral method serves as a test case. Problem 4 (without the axial conduction effect) is the well-known Graetz problem [5], whose solution is known. In this work we further generate the solution for smaller Péclét numbers.

In our present analysis, we test the assumption of 2nd, 3rd and 4th order polynominal approximations.

To form these profiles we take advantage of the following conditions which apply to the temperature (or concentration) profiles:

$$\eta = \frac{y}{l} = 0 \qquad \theta = \frac{T - T_0}{T_w - T_0} = 1 \tag{10}$$

$$\eta = 1 \qquad \theta = \theta_l\left(\frac{x}{l}\right) = \theta_l(\xi) \tag{11}$$

$$\eta = 1 \qquad \frac{\partial \theta}{\partial \eta} = 0$$

$$\eta = 0 \qquad \frac{\partial^2 \theta}{\partial \eta^2} = 0 \text{(in channel)};$$

$$\frac{\partial}{\partial \eta}(1 - \eta)\frac{\partial \theta}{\partial \eta} = 0 \text{(in pipes)} \tag{12}$$

$$\eta = 0 \qquad \frac{\partial^3 \theta}{\partial \eta^3} = 0 \text{(in channel)};$$

$$\frac{\partial}{\partial \eta}\left[\frac{\partial}{\partial \eta}(1 - \eta)\frac{\partial \theta}{\partial \eta}\right] = 0 \text{(in pipes)} \tag{13}$$

under such conditions the following profiles were obtained:

Second order polynomial

$$\theta = 1 - (1 - \theta_l)(2\eta - \eta^2) \quad \text{(in channel and pipe).} \tag{14}$$

Third order polynomials

$$\theta = 1 - (1 - \theta_l)(\tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3) \quad \text{(in channel)} \tag{15a}$$

$$\theta = 1 - (1 - \theta_l)(\tfrac{6}{5}\eta + \tfrac{3}{5}\eta^2 - \tfrac{4}{5}\eta^3) \quad \text{(in pipe).} \tag{15b}$$

Fourth order polynomial

$$\theta = 1 - (1 - \theta_l)(\tfrac{4}{3}\eta - \tfrac{1}{3}\eta^4) \quad \text{(in channel)} \tag{16a}$$

$$\theta = 1 - (1 - \theta_l)(\tfrac{12}{13}\eta + \tfrac{6}{13}\eta^2 + \tfrac{4}{13}\eta^3 - \tfrac{9}{13}\eta^4) \quad \text{(in pipe).} \tag{16b}$$

The fully developed velocity profiles considered were:

$$u = \tfrac{3}{2}\bar{u}(1 - \eta^2) \text{(in channel)} \tag{17a}$$

$$u = 2\bar{u}(2\eta - \eta^2) \text{(in pipe).} \tag{17b}$$

From hereon the solution is straightforward. Substituting equation (14) to (17) into the heat integral equation (7) or (8) results in a second-order linear differential equation for $\theta_l(x)$. The boundary conditions for $\theta_l(x)$ are

$$\begin{aligned} x &= 0 \qquad \theta_l = 0 \\ x &= \infty \qquad \theta_l = 1. \end{aligned} \tag{18}$$

We have here four cases and considering the fact that for each problem we obtain a solution using three assumed 2nd, 3rd and 4th order polynomials, we deal with 12 solutions. However all these problems can be treated simultaneously noticing that the differences between the different problems are only with respect to constant coefficients.

Thus the energy equations (7) and (8) take the general form

$$\frac{d^2\theta_l}{d\xi^2} - c_1 Pe\frac{d\theta_l}{d\xi} - c_2\theta_l = -c_2 \tag{19}$$

and the solution is:

$$\theta_l = 1 - \exp\left[\left[\left\{\frac{Pe}{2}c_1 - \sqrt{\left[\left(\frac{Pe}{2}c_1\right)^2 + c_2\right]}\right\}\xi\right]\right] \tag{20}$$

where c_1 and c_2 for the various problems and assumed temperature (concentration) profiles are given in Table 1.

For large Pe equation (20) reduces into

$$\theta_l \simeq 1 - \exp\left[-\frac{c_2}{c_1}\frac{\xi}{Pe}\right]. \tag{21}$$

HEAT TRANSFER

The solution yields $\theta_l(x)$. We may compare θ_l with the exact solution (when available) and indeed such a comparison shows quite good agreement. Nevertheless our interest lies in comparing heat-transfer characteristics.

The net amount of heat transfered to the fluid is:

$$Q = \rho c_p \bar{u} A(\bar{T} - T_0) \tag{22}$$

where the average temperature $\bar{T}$ is defined as usual (see nomenclature).

Unlike the case of negligible axial conduction our definition of the average temperature is not synonymous to the usually so called "mixing cup" temperature. In fact the whole concept of the "mixing cup" temperature does not fit the

Table 1. The constants c_1, c_2 *and* c_3

	Incline flow					
	Slug flow			Fully developed flow		
	2nd	3rd	4th	2nd	3rd	4th
c_1	0·5000	0·5000	0·5000	0·4125	0·370	0·3929
c_2	3·0000	2·4	2·2222	3·000	2·4000	2·2222
c_3	0·6667	0·6250	0·6000	0·5500	0·4625	0·4714

	Pipe flow					
	Slug flow			Fully developed flow		
	2nd	3rd	4th	2nd	3rd	4th
c_1	0·500	0·5000	0·5000	0·6667	0·6939	0·7099
c_2	8·000	5·7143	5·000	8·00	5·7143	5·000
c_3	0·500	0·4200	0·3692	0·6667	0·5829	0·5242

case with axial diffusion. This is because it is impossible to apply a collecting cup "test" at any cross section of the channel without interfering with the temperature and flux distribution upstream (namely, we "distort" the downstream boundary condition and change our problem). In addition we may mention that for negligible axial diffusion the product $\rho c_p \bar{u} \bar{T} A$ (for constant physical properties) represents the energy flux in the axial direction. This is not the case with axial diffusion where it is required to add the conducted axial energy flux.

Thus the meaning of our definition for the average temperature is simply the sum of the local measured temperature weighted by the local velocities. This expresses exactly the heat absorbed by the fluid (per unit time) which causes the rise in its temperature.*

* Notice also that when the temperature is uniform with respect to y but is a function of x the average temperature as calculated equals this uniform temperature. This may not be the case if we define, for example, a new modified bulk temperature as mention in the Appendix of [1].

Proceeding as usual we define a heat-transfer coefficient h as:

$$h \equiv \frac{dQ}{(T_w - \bar{T})\,dx} \tag{23}$$

which leads to the well-known heat-transfer formula

$$Q = \bar{h} s \,\Delta T_{\ln} = \bar{h} s \frac{(T_w - T_0) - (T_w - \bar{T})}{\ln\left[(T_w - T_0)/(T_w - \bar{T})\right]} \tag{24}$$

where

$$\bar{h} = \frac{1}{x}\int_0^x h\,dx. \tag{25}$$

Thus, heat-transfer calculations for the case with axial diffusion is treated here similarly to the case where axial diffusion is ignored. The differences, however, lie in the fact that for Péclét number less than infinity heat is also conducted along the axial direction. This obviously causes heat losses to the inlet boundary at $x = 0$. The amount of heat Q in equations (22)–(24) is, thus, only the net heat absorbed by the fluid used for raising its temperature. This amount is less than the amount required when axial diffusion is absent. It is, therefore, expected that the average heat-transfer coefficient $\bar{h}$ (or Nu) for the case where axial diffusion is considered to be less than the one calculated assuming Péclét number of infinity. Thus our analysis predicts:

$$Nu_l = \frac{\bar{h} l}{k} = -\frac{\rho c_p \bar{u} l^2}{kx}\ln\frac{T_w - \bar{T}}{T_w - T_0} = -\frac{Pe}{2\xi}\ln(1 - \bar{\theta})\ \text{(in channel)} \tag{26a}$$

$$Nu_d = \frac{\bar{h} d}{k} = -\frac{\rho c_p \bar{u} l^2}{kx}\ln\frac{T_w - \bar{T}}{T_w - T_0} = -\frac{Pe}{2\xi}\ln(1 - \bar{\theta})\ \text{(in pipes)}. \tag{26b}$$

$\bar{\theta}$ is given by

$$\bar{\theta} = 1 - c_3(1 - \theta_l) \tag{27}$$

where c_3 is also reported in Table 1 for all cases considered.

For large ξ equations (26) reduces into:

$$Nu = -\frac{Pe}{2}\left\{Pe\frac{c_1}{2} - \sqrt{\left[Pe^2\left(\frac{c_1}{2}\right)^2 + c_2\right]}\right\}. \tag{28}$$

Equation (28) yields $\overline{Nu}_t$ for channel flow and $\overline{Nu}_d$ for pipe flow.

RESULTS

As in most practical cases where axial diffusion is ignored, we are also mainly interested in the value of $\overline{Nu}$ (or $\overline{Sh}$) numbers for large x.* This fact is also consistent with our approximations which specifies the region for small x as of limited accuracy.

Table 2 and 3 reports the results of $\overline{Nu}$ at large ξ for all cases considered. These tables contain the solutions using 2nd, 3rd and 4th order polynomials. For slug flows exact solutions are easily derived (as shown in Appendix A) and their results for the $\overline{Nu}$ number is also tabulated. For the incline plane problem with fully developed velocity profile the solution is available in [1].* For the fully developed pipe flow (Graetz problem) only a solution for large Péclét number is available (axial conduction ignored). Comparison between the exact and integral method solution, when available, shows that usually a 3rd order polynomial approximation is our best choice. Using third order temperature profile results usually in errors of less than 5 per cent. Only for the Graetz problem the choice of 4th order polynomial yields more accurate results (error of 4 per cent) than the 3rd order polynomial which yields a maximum error of 12 per cent.

As a general rule, however, we tend to conclude that we expect a third order polynomial approximation to be our overall best choice

Table 2. Nu_t *for incline flow*

	Slug flow				Fully developed flow			
Pe	Exact	2nd	3rd	4th	Exact	2nd	3rd	4th
0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
0·1	0·0773	0·0854 (+10)	0·0762 (−1)	0·0733 (−5)	0·0777	0·0856 (+10)	0·0765 (−2)	0·0736 (−5)
0·2	0·1521	0·1683 (+11)	0·1500 (−1)	0·1442 (−5)	0·1536	0·1691 (+10)	0·1513 (−1)	0·1452 (−5)
0·5	0·3627	0·4029 (+11)	0·3573 (−1)	0·3427 (−6)	0·3715	0·4080 (+10)	0·3649 (−2)	0·3489 (−6)
1·0	0·6703	0·7500 (+12)	0·6596 (−2)	0·6308 (−6)	0·7026	0·7690 (+9)	0·6876 (−2)	0·6536 (−7)
5·0	1·894	2·215 (+17)	1·852 (−2)	1·739 (−8)	2·286	2·461 (+8)	2·198 (−4)	2·008 (−12)
10·0	2·263	2·707 (+20)	2·205 (−3)	2·054 (−9)	2·948	3·154 (+7)	2·815 (−5)	2·508 (−15)
100·0	2·465	2·996 (+22)	2·398 (−3)	2·220 (−10)		3·630	3·238	2·824
∞	2·467	3·000 (+22)	2·400 (−3)	2·222 (−10)	3·414	3·636 (+7)	3·243 (−5)	2·828 (−17)

Note: Figures in parentheses are deviations (%) from the exact calculation

* Usually of the order $x/l > \frac{1}{10}\,Pe$.

* In this reference $\overline{Nu}$ is defined differently. However for large ξ the numerical value for $\overline{Nu}$ reported by this reference agrees with our definition.

Table 3. $\overline{\mathrm{Nu}}_d$ *for pipe flow*

	Slug flow				Fully developed flow			
Pe	Exact	2nd	3rd	4th	Exact	2nd	3rd	4th
0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
0·1	0·1190	0·1402 (+18)	0·1183 (−1)	0·1106 (−7)		0·1398	0·1178	0·1100
0·2	0·2355	0·2779 (+18)	0·2341 (−1)	0·2187 (−7)		0·2763	0·2322	0·2166
0·5	0·5708	0·6765 (+19)	0·5672 (−1)	0·5286 (−7)		0·6667	0·5558	0·5164
1·0	1·084	1·295 (+19)	1·077 (−1)	1·000 (−8)		1·257	1·034	0·9546
5·0	3·651	4·606 (+26)	3·619 (−1)	3·279 (−10)		4·041	3·047	2·700
10·0	4·844	6·375 (+32)	4·795 (−1)	4·271 (−12)		5·191	3·719	3·228
100·0	5·770	7·975 (+38)	5·701 (−1)	4·990 (−14)		5·990	4·113	3·518
∞	5·783	8·000 (+38)	5·714 (−1)	5·000 (−14)	3·66	6·000 (+64)	4·118 (+12)	3·522 (−4)

Note: Figures in parentheses are deviations (%) from the exact solution.

with expected accuracy less than 12 per cent. One may notice, also, that the maximum error occurs usually for large Péclét numbers. The error in our calculation decreases usually for smaller Péclét numbers and is zero for $Pe = 0$.

SUMMARY AND CONCLUSIONS

The integral method, so far being used for convection problems without axial diffusion, is also applied to problems with axial diffusion. The integral heat-balance equation for this case is given by equation (7) for the case of channel flow and by equation (8) for the axi-symmetrical flow.

These equations, however, are still difficult to solve for $\delta(x)$ (for the semi-infinite flow region or near the starting point of the thermal disturbance) as it is usually done when axial diffusion is neglected. Yet, general overall heat-transfer characteristics for flow in channels or pipes is obtained analytically, and quite easily by the aforementioned presented method. For this purpose it is recommended to use a 3rd order polynomial for the temperature/concentration profiles. The expected accuracy is better than 12 per cent.

REFERENCES

1. Z. Rotem and J. E. Neilson, Exact solution for diffusion to flow down an incline, *Can. J. Chem. Engng* **47**, 341–346 (1969).
2. M. J. Lighthill, Initial development of diffusion in Poiseuille flow, *J. Inst. Math. Applics* **2**, 97–108 (1966).
3. Kh. Boyadzhiev, V. Levich and V. Krylov, Improving of the theory of convection diffusion, *Int. Chem. Engng* **8**, 393–396 (1968).
4. C. Hsu, An exact mathematical solution for entrance-region laminar heat transfer with axial conduction, *Appl. Sci. Res.* **A17**, 359–376 (1967).
5. J. G. Knudsen and D. L. Katz, *Fluid Dynamics and Heat Transfer*, p. 368. McGraw-Hill, New York (1958).
6. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, p. 493. Oxford University Press, London (1959).

APPENDIX

Exact Solutions with Axial Diffusion-Slug Flow

For incline plane flow equation (1) takes the form

$$\frac{\partial \theta'}{\partial \xi} = \frac{2}{Pe}\left[\frac{\partial^2 \theta'}{\partial \xi^2} + \frac{\partial^2 \theta'}{\partial \eta'^2}\right] \quad (A.1)$$

where for pipe flow it is written as

$$\frac{\partial \theta'}{\partial \xi} = \frac{2}{Pe}\left[\frac{\partial^2 \theta'}{\partial \xi^2} + \frac{\partial^2 \theta'}{\partial \eta'^2} + \frac{1}{\eta'}\frac{\partial \theta'}{\partial \eta'}\right]. \quad (A.2)$$

Note that for this solution we use the primed co-ordinate system in Fig. 1, (x', y') and $\theta' = (T - T_w)/(T_0 - T_w) = 1 - \theta$. The boundary conditions are:

$$\begin{aligned} \xi = 0: &\quad \theta' = 1 \\ \xi \to \infty: &\quad \theta = 0 \,(\text{or } \partial\theta'/\partial\xi = 0) \\ \eta' = 0: &\quad \partial\theta'/\partial\eta' = 0 \\ \eta' = 1: &\quad \theta' = 0. \end{aligned} \quad (A.3)$$

A solution is obtained via separation of variables. For incline plane the local dimensionless temperature is:

$$\theta' = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\left(\frac{2n-1}{2}\pi\eta'\right)\exp\left(-\frac{2\lambda_n^2\xi}{Pe}\right) \quad (A.4)$$

where

$$\lambda_n^2 = -\frac{Pe^2}{8} + \sqrt{\left[\frac{Pe^4}{64} + Pe^2(2n-1)^2\frac{\pi^2}{16}\right]}. \quad (A.5)$$

For pipe flow,

$$\theta' = 2\sum_{n=1}^{\infty}\frac{1}{\alpha_n}\frac{J_0(\alpha_n\eta')}{J_1(\alpha_n)}\exp\left(-\frac{2\lambda_n^2\xi}{Pe}\right) \quad (A.6)$$

where

$$\lambda_n^2 = -\frac{Pe^2}{8} + \sqrt{\left(\frac{Pe^4}{64} + Pe^2\frac{\alpha_n^2}{4}\right)}. \quad (A.7)$$

The constants α_n are roots of $J_0(\alpha_n) = 0$ where the first six values are reported in [6]. J_0 and J_1 designate Bessel's function of the first kind of zero and first order respectively.

For incline plane the average temperature is

$$\bar{\theta}' = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\exp\left(-\frac{2\lambda_n^2\xi}{Pe}\right). \quad (A.8)$$

For pipe flow it is given by

$$\bar{\theta}' = 4\sum_{n=1}^{\infty}\frac{1}{\alpha_n^2}\exp\left(-\frac{2\lambda_n^2\xi}{Pe}\right). \quad (A.9)$$

Using equations (26a) and (26b) yields average Nusselt numbers for $\xi \to \infty$ as,

$$\overline{Nu}_l = \lambda_1^2 = -\frac{Pe^2}{8} + \sqrt{\left(\frac{Pe^4}{64} + Pe^2\frac{\pi^2}{16}\right)} \text{(incline plane)} \quad (A.10)$$

$$\overline{Nu}_d = \lambda_1^2 = -\frac{Pe^2}{8} + \sqrt{\left(\frac{Pe^4}{64} + Pe^2\frac{\alpha_1^2}{4}\right)} \text{(pipe flow)} \quad (A.11)$$

where $\bar{u}_1 = 2{\cdot}4048$.

Numerical values of equations (A.10) and (A.11) are reported in Table 2.

APPLICATION DE LA MÉTHODE INTÉGRALE À DES ÉCOULEMENTS AVEC DIFFUSION AXIALE

Résumé—Ce travail étend l'application de la méthode intégrale à des problèmes de diffusion axiale habituellement ignorée dans les écoulements convectifs. On démontre que les caractéristiques de transfert thermique (ou massique) peuvent être obtenues sous une forme analytique et avec une précision satisfaisante. On donne les résultats relatifs au problème de Graetz et à d'autres problèmes de diffusion axiale.

DIE ANWENDUNG DER INTEGRALMETHODE AUF STRÖMUNGEN MIT AXIALER DIFFUSION

Zusammenfassung—Diese Arbeit erweitert die Anwendung der Integralmethode auf Probleme mit axialer Diffusion, deren Vorhandensein gewöhnlich in konvektiven Strömungsproblemen unberücksichtigt bleibt. Es wird dargelegt, dass Wärme- (oder Stoff-)Übertragungseigenschaften in einer geschlossenen Darstellungsweise mit zufriedenstellender Genauigkeit gewonnen werden können. Ergebnisse für das Graetz-Problem und andere Probleme mit axialer Diffusion sind angegeben.

ИСПОЛЬЗОВАНИЕ ИНТЕГРАЛЬНОГО МЕТОДА ДЛЯ ТЕЧЕНИЙ С АКСИАЛЬНОЙ ДИФФУЗИЕЙ

Аннотация—В работе проводится обобщение интегрального метода для задач с аксиальной диффузией, которой в задачах о конвективных течениях обычно пренебрегают. Показано, что тепло- (или массо-) обменные характеристики могут быть получены в замкнутом виде с удовлетворительной точностью. Приведены результаты решений задачи Грэтца и других задач при аксиальной диффузии.